

Relations

Chapter 9

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Chapter Summary

Relations and
Their
Properties

n -ary Relations
and Their
Applications

Representing
Relations

Closures of
Relations

Equivalence
Relations

Partial
Orderings

RELATIONS AND THEIR PROPERTIES

Section 9.1

Section Summary

Relations and Functions

Properties of Relations

- Reflexive and Irreflexive Relations
- Symmetric and Antisymmetric Relations
- Transitive Relations

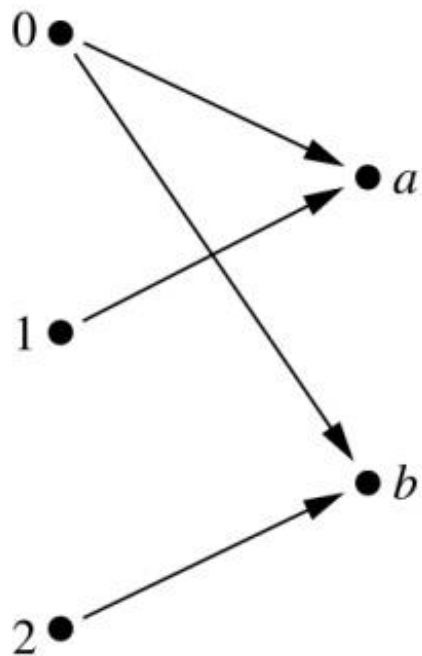
Combining Relations

Binary Relations

Definition: A **binary relation** R from a set A to a set B is a subset $R \subseteq A \times B$.

Example:

- Let $A = \{0,1,2\}$ and $B = \{a, b\}$
- $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B .
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of B is related to each element of A .

Binary Relation on a Set

Definition: A binary relation R on a set A is a subset of $A \times A$ or a relation from A to A .

Example:

- Suppose that $A = \{a, b, c\}$. Then $R = \{(a, a), (a, b), (a, c)\}$ is a relation on A .
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation $R = \{(a, b) | a \text{ divides } b\}$ are $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$

Binary Relation on a Set

Question: How many relations are there on a set A ?

Solution: Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$. Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$. Therefore, there are 2^{n^2} relations on a set A .

Binary Relations on a Set

Example: Consider these relations on the set of integers:

$$R_1 = \{(a, b) | a \leq b\}, R_2 = \{(a, b) | a > b\}, R_3 = \{(a, b) | a = b \text{ or } a = -b\}, \\ R_4 = \{(a, b) | a = b\}, R_5 = \{(a, b) | a = b + 1\}, R_6 = \{(a, b) | a + b \leq 3\}.$$

Which of these relations contain each of the pairs

$$(1,1), (1,2), (2,1), (1, -1) \text{ and } (2,2)?$$

Solution: Checking the conditions that define each relation, we see that the pair $(1,1)$ is in R_1, R_3, R_4 and R_6 ; $(1,2)$ is in R_1 and R_6 ; $(2,1)$ is in R_2, R_5 , and R_6 ; $(1, -1)$ is in R_2, R_3 and R_6 ; $(2,2)$ is in R_1, R_3 and R_4 .

Reflexive Relations

Definition: R is **reflexive** iff $(a, a) \in R$ for every element $a \in A$. Written symbolically, R is reflexive if and only if

$$\forall x [x \in A \rightarrow (x, x) \in R]$$

The following relations are reflexive:

$$R_1 = \{(a, b) | a \leq b\}$$

$$R_3 = \{(a, b) | a = b \text{ or } a = -b\}$$

$$R_4 = \{(a, b) | a = b\}$$

The following relations are not reflexive:

$$R_2 = \{(a, b) | a > b\}$$

$$R_5 = \{(a, b) | a = b + 1\}$$

$$R_6 = \{(a, b) | a + b \leq 3\}$$

- If $A = \emptyset$ then the empty relation is reflexive vacuously. That is the empty relation on an empty set is reflexive!

Reflexive Relations

Example: How many reflexive relations are there on a set with n elements?

Solution: A relation R on a set A is a subset of $A \times A$. Consequently, a relation is determined by specifying whether each of the n^2 ordered pairs in $A \times A$ is in R .

However, if R is reflexive,

- each of the n ordered pairs (a, a) for $a \in A$ must be in R .
- Each of the other $n(n - 1)$ ordered pairs of the form (a, b) , where $a \neq b$, may or may not be in R . Hence, by the product rule for counting, there are $2^{n(n-1)}$ reflexive relations.

Irreflexive Relations

Definition: A relation R on the set A is **irreflexive** if for every $a \in A$, $(a, a) \notin R$. That is, R is irreflexive if no element in A is related to itself.

$$\forall x [x \in A \rightarrow (x, x) \notin R]$$

The following relations are irreflexive:

$$R_2 = \{(a, b) | a > b\}$$

$$R_5 = \{(a, b) | a = b + 1\}$$

The following relations are not irreflexive:

$$R_1 = \{(a, b) | a \leq b\}$$

$$R_3 = \{(a, b) | a = b \text{ or } a = -b\}$$

$$R_4 = \{(a, b) | a = b\}$$

$$R_6 = \{(a, b) | a + b \leq 3\}$$

Symmetric Relations

Definition: R is **symmetric** iff $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$. Written symbolically, R is symmetric if and only if

$$\forall x \forall y [(x, y) \in R \rightarrow (y, x) \in R]$$

The following relations are symmetric:

$$R_3 = \{(a, b) | a = b \text{ or } a = -b\}$$

$$R_4 = \{(a, b) | a = b\}$$

$$R_6 = \{(a, b) | a + b \leq 3\}$$

The following relations are not symmetric:

$$R_1 = \{(a, b) | a \leq b\}$$

$$R_2 = \{(a, b) | a > b\}$$

$$R_5 = \{(a, b) | a = b + 1\}$$

Antisymmetric Relations

Definition: A relation R on a set A such that for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called **antisymmetric**. Written symbolically, R is antisymmetric if and only if

$$\forall x \forall y [(x, y) \in R \wedge (y, x) \in R \rightarrow x = y]$$

The following relations are antisymmetric: The following relations are not antisymmetric:

$$R_1 = \{(a, b) | a \leq b\}$$

$$R_2 = \{(a, b) | a > b\}$$

$$R_4 = \{(a, b) | a = b\}$$

$$R_5 = \{(a, b) | a = b + 1\}$$

$$R_3 = \{(a, b) | a = b \text{ or } a = -b\}$$

$$R_6 = \{(a, b) | a + b \leq 3\}$$

Transitive Relations

Definition: A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y \forall z [(x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R]$$

The following relations are transitive:

$$R_1 = \{(a, b) | a \leq b\}$$

$$R_2 = \{(a, b) | a > b\}$$

$$R_3 = \{(a, b) | a = b \text{ or } a = -b\}$$

$$R_4 = \{(a, b) | a = b\}$$

The following relations are not transitive:

$$R_5 = \{(a, b) | a = b + 1\}$$

$$R_6 = \{(a, b) | a + b \leq 3\}$$

Is the “divides” relation on the set of positive integers reflexive? Irreflexive? Symmetric? Antisymmetric? Transitive?

How many relations are there on a set with n –elements that are

- reflexive ? $2^{n(n-1)}$
- irreflexive? $2^{n(n-1)}$
- symmetric? $2^{n(n+1)/2}$
- antisymmetric? $2^n 3^{n(n-1)/2}$
- reflexive and symmetric? $2^{n(n-1)/2}$
- neither reflexive nor irreflexive? $2^{n^2} - 2 \times 2^{n(n-1)}$

Combining Relations

- Given two relations R_1 and R_2 , we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$ and $R_1 \oplus R_2$.

Example: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1), (2,2), (3,3)\}$ and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\}$$

$$R_1 - R_2 = \{(2,2), (3,3)\}$$

$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$$

$$R_1 \oplus R_2 = \{(1,2), (1,3), (1,4), (2,2), (3,3)\}$$

Composition

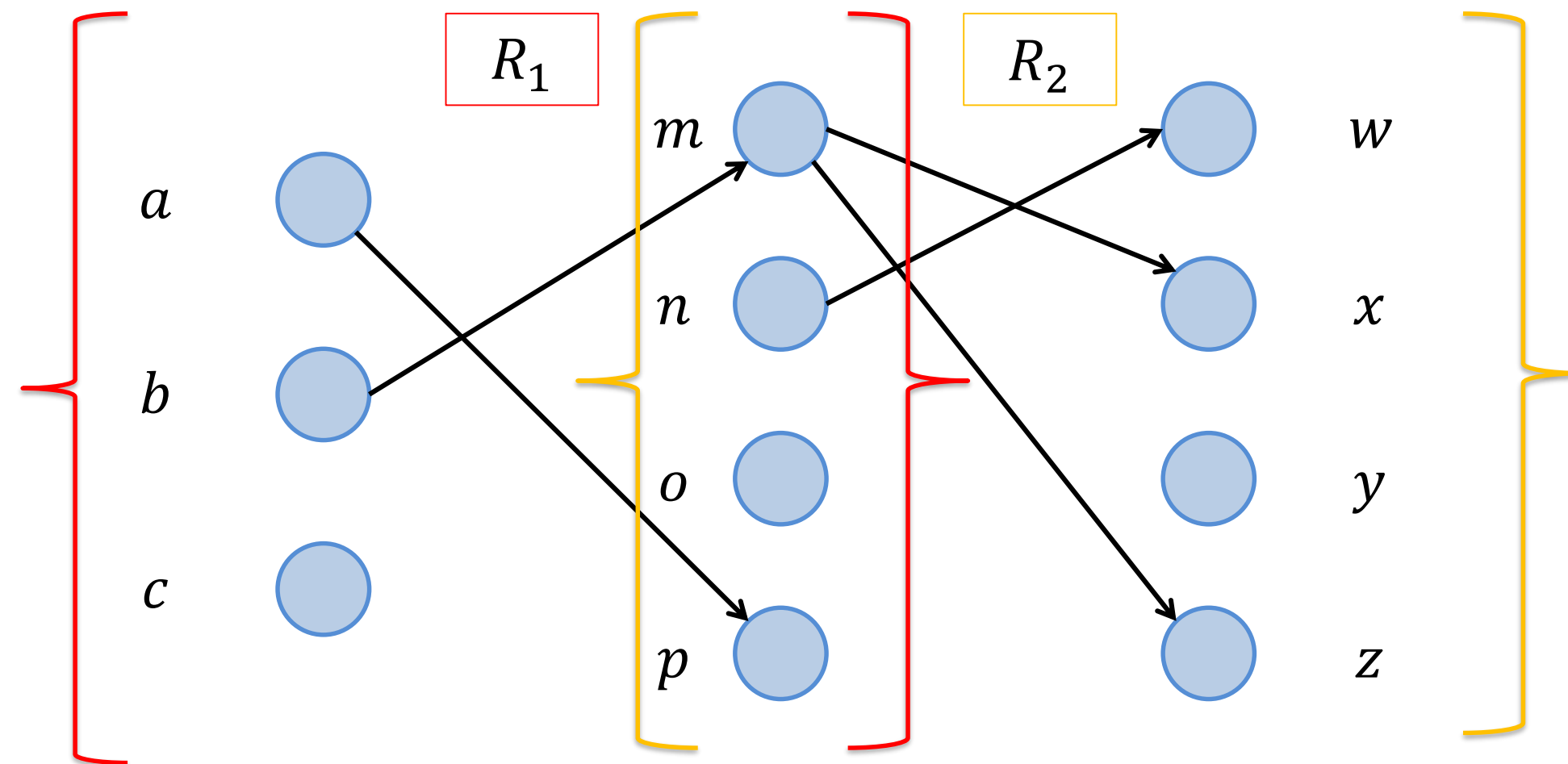
Definition: Suppose

- R_1 is a relation from a set A to a set B .
- R_2 is a relation from a set B to a set C .

Then the **composition (or composite)** of R_2 with R_1 , is a relation from A to C where

- if (x, y) is a member of R_1 and (y, z) is a member of R_2 , then (x, z) is a member of $R_2 \circ R_1$.

Representing the Composition of a Relation



$$R_2 \circ R_1 = \{(b, x), (b, z)\}$$

Powers of a Relation

Definition: Let R be a binary relation on A . Then the **powers** R^n of the relation R can be defined inductively by:

- Basis Step: $R^1 = R$
- Inductive Step: $R^{n+1} = R^n \circ R$

The powers of a transitive relation are subsets of the relation. This is established by the following theorem.

Powers of a Relation

Theorem: The relation R on a set A is transitive iff $R^n \subseteq R$ for $n = 1, 2, 3 \dots$

Proof: We first prove the “if” part of the theorem. We suppose that $R^n \subseteq R$ for $n = 1, 2, 3, \dots$. In particular, $R^2 \subseteq R$. To see that this implies R is transitive, note that if $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition, $(a, c) \in R^2$. Because $R^2 \subseteq R$, this means that $(a, c) \in R$. Hence, R is transitive.

We will use mathematical induction to prove the only if part of the theorem. Note that this part of the theorem is trivially true for $n = 1$.

Powers of a Relation

Assume that $R^n \subseteq R$, where n is a positive integer. This is the inductive hypothesis. To complete the inductive step we must show that this implies that R^{n+1} is also a subset of R .

To show this, assume that $(a, b) \in R^{n+1}$. Then, because $R^{n+1} = R^n \circ R$, there is an element x with $x \in A$ such that $(a, x) \in R$ and $(x, b) \in R^n$. The inductive hypothesis, namely, that $R^n \subseteq R$, implies that $(x, b) \in R$.

Furthermore, because R is transitive, and $(a, x) \in R$ and $(x, b) \in R$, it follows that $(a, b) \in R$. This shows that $R^{n+1} \subseteq R$, completing the proof.

n-ARY RELATIONS AND THEIR APPLICATIONS

Section 9.2

n -ary Relations

Definition: Let $A_1, A_2, \dots, A_n$ be sets. An n -ary relation on these sets is a subset of $A_1 \times A_2 \times \dots \times A_n$. The sets $A_1, A_2, \dots, A_n$ are called the **domains** of the relation, and n is called its **degree**.

Example: Let R be the relation on $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^+$ consisting of triples (a, b, m) , where a, b and m are integers with $m \geq 1$ and $a \equiv b \pmod{m}$. Then $(8, 2, 3)$, $(-1, 9, 5)$ and $(14, 0, 7)$ all belong to R , but $(7, 2, 3)$, $(-2, -8, 5)$ and $(11, 0, 6)$ do not belong to R . This relation has degree 3 and its first two domains are the set of all integers and its third domain is the set of positive integers.

Databases and Relations

- Various methods for representing databases have been developed. We discuss the **relational data model**, based on the concept of a relation.
- A database consists of **records**, which are n -tuples, made up of **fields**.
- Relations used to represent databases are also called **tables**.
- A domain of an n -ary relation is called a **primary key** when the value of the n -tuple from this domain determines the n -tuple.
- When the values of a set of domains determine an n –tuple in a relation, the Cartesian product of these domains is called a **composite key**.

Databases and Relations

Example1: Which domains are primary keys for the n -ary relation, assuming that no n -tuples will be added in the future?

Example2: Is the Cartesian product of the domain of major fields of study and the domain of GPAs a composite key for the n -ary relation, assuming that no n -tuples are ever added?

Solution1:

Student_name,

ID_number.

Solution2: Yes.

Student_name	ID_number	Major	GPA
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Good friend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Databases and Relations

- Records are often added to or deleted from databases. So the property that a domain is a primary key is time-dependent. Consequently, a primary key should be chosen that remains one whenever the database is changed.
- The current collection of n -tuples in a relation is called the **extension** of the relation. The more permanent part of a database, including the name and attributes of the database, is called its **intension**.
- When selecting a primary key, the goal should be to select a key that can serve as a primary key for all possible extensions of the database.
- To do this, it is necessary to examine the intension of the database to understand the set of possible n –tuples that can occur in an extension.

Operations on n -ary Relations

Definition: Let R be an n -ary relation and C a condition that elements in R may satisfy. Then the **selection operator** s_C maps the n -ary relation R to the n -ary relation of all n -tuples from R that satisfy the condition C .

Definition: The **projection** $P_{i_1 i_2 \dots i_m}$ where $i_1 < i_2 < \dots < i_m$, maps the n -tuple $(a_1, a_2, \dots, a_n)$ to the m -tuple $(a_{i_1}, a_{i_2}, \dots, a_{i_m})$, where $m \leq n$.

Definition: Let R be a relation of degree m and S a relation of degree n . The **join** $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p)$ -tuples $(a_1, a_2, \dots, a_{m-p}, c_1, c_2, \dots, c_p, b_1, b_2, \dots, b_{n-p})$, where the m -tuple $(a_1, a_2, \dots, a_{m-p}, c_1, c_2, \dots, c_p)$ belongs to R and the n -tuple $(c_1, c_2, \dots, c_p, b_1, b_2, \dots, b_{n-p})$ belongs to S .

REPRESENTING RELATIONS

Section 9.3

Section Summary

Representing
Relations using
Matrices

Representing
Relations using
Digraphs

Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- Suppose R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$.
 - The elements of the two sets can be listed in any particular arbitrary order.

When $A = B$, we use the same ordering.

- The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$$

- The matrix representing R has a 1 as its (i, j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j .

Examples of Representing Relations Using Matrices

Example: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a, b) if $a \in A, b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?

Solution: Because $R = \{(2,1), (3,1), (3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Examples of Representing Relations Using Matrices

Example: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$

Matrices of Relations on Sets

- R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.
- R is an irreflexive relation, all the elements on the main diagonal of M_R are equal to 0.
- R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$. $M_R = M_R^t$.
- R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & \ddots & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} & & 1 \\ & \diagdown & \\ 1 & & \\ & \diagup & 0 \\ & 0 & \end{bmatrix}$$

(a) Symmetric

$$\begin{bmatrix} & & 1 & & \\ & \diagdown & & 0 & \\ 0 & & \diagup & & \\ & 0 & & \diagup & 0 \\ & & 0 & 1 & \end{bmatrix}$$

(b) Antisymmetric

Example of a Relation on a Set

Example: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, irreflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements are equal to 1, R is reflexive and not irreflexive.

Because M_R is symmetric, R is symmetric.

Because both $m_{1,2}$ and $m_{2,1}$ are 1, R is not antisymmetric.

Boolean Operations

- The Boolean operations join and meet can be used to find the matrices representing the union and the intersection of two relations.
- Suppose that R_1 and R_2 are relations on a set A represented by the matrices M_{R_1} and M_{R_2} , respectively. The matrix representing the union of these relations has a 1 in the positions where either M_{R_1} or M_{R_2} has a 1. The matrix representing the intersection of these relations has a 1 in the positions where both M_{R_1} and M_{R_2} have a 1.
- Thus, the matrices representing the union and intersection of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}, \quad M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

Union and Intersection

Example: Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution: The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix for the Composite of Relations

- This matrix can be found using the Boolean product of the matrices for these relations. Suppose that R is a relation from A to B and S is a relation from B to C . Suppose that A, B and C have m, n and p elements, respectively. Let the zero-one matrices for $S \circ R, R$ and S be $M_{S \circ R} = [t_{ij}], M_R = [r_{ij}]$ and $M_S = [s_{ij}]$, respectively.
- The ordered pair (a_i, c_j) belongs to $S \circ R$ if and only if there is an element b_k such that (a_i, b_k) belongs to R and (b_k, c_j) belongs to S . It follows that $t_{ij} = 1$ if and only if $r_{ik} = s_{kj} = 1$ for some k . This means that

$$M_{S \circ R} = M_R \odot M_S$$

Matrix for the Composite of Relations

Example: Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution: The matrix for $S \circ R$ is

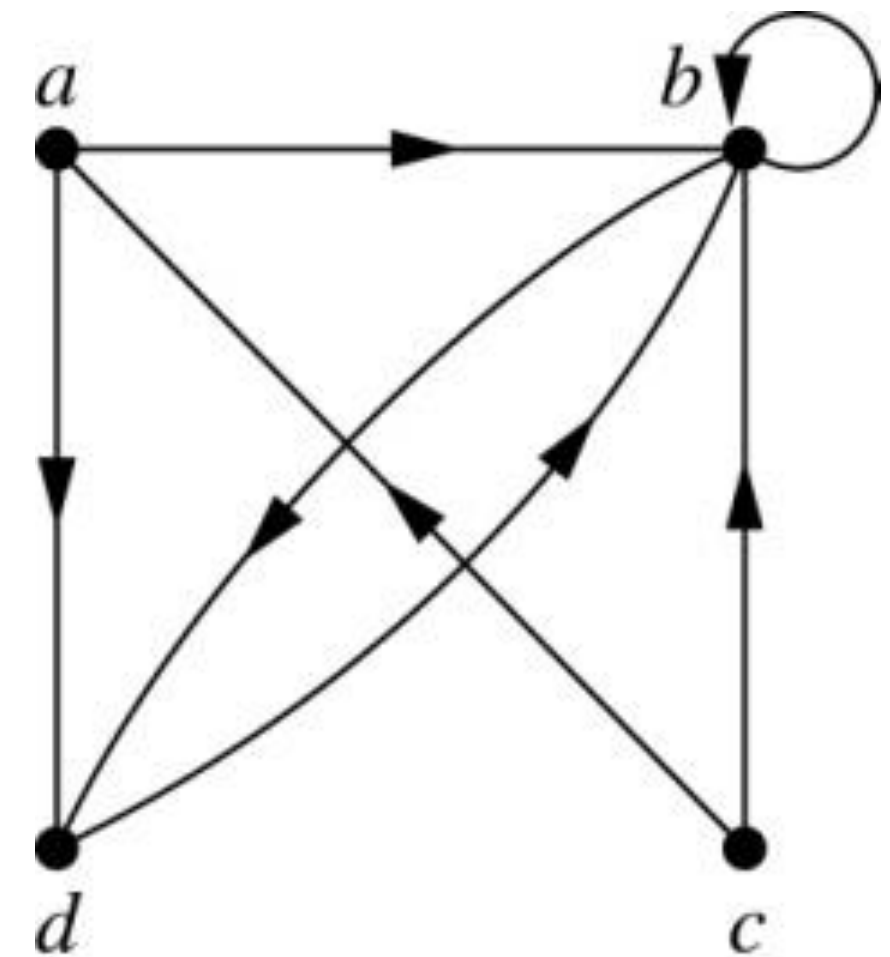
$$M_{S \circ R} = M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Representing Relations Using Digraphs

Definition: A **directed graph**, or **digraph**, consists of a set V of **vertices** (or **nodes**) together with a set E of ordered pairs of elements of V called **edges** (or **arcs**). The vertex a is called the **initial vertex** of the edge (a, b) , and the vertex b is called the **terminal vertex** of this edge.

- An edge of the form (a, a) is called a **loop**.

Example: A drawing of the directed graph with vertices a, b, c and d , and edges (a, b) , (a, d) , (b, b) , (b, d) , (c, a) , (c, b) and (d, b) is shown here.



Examples of Digraphs Representing Relations

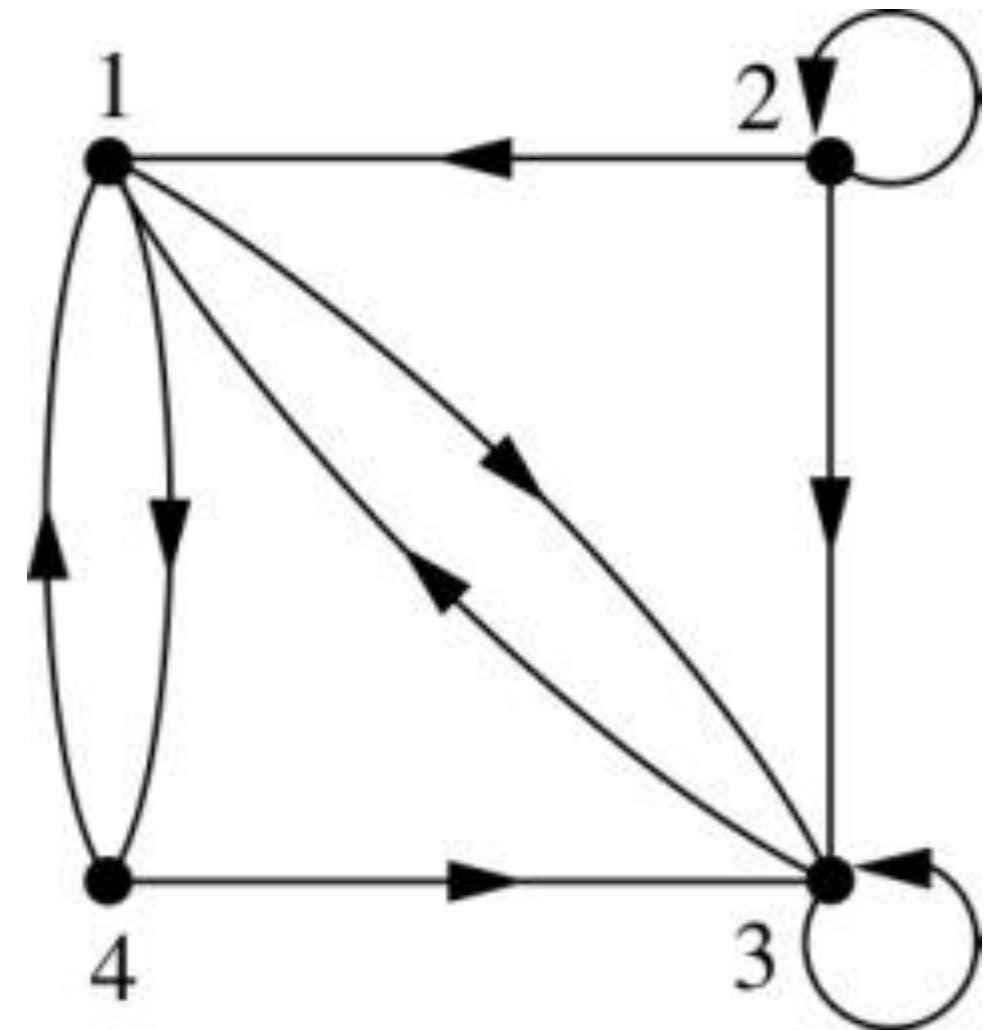
Example: What are the ordered pairs in the relation represented by this directed graph?

Solution: The ordered pairs in the relation are

$(1, 3), (1, 4), (2, 1),$

$(2, 2), (2, 3), (3, 1),$

$(3, 3), (4, 1), (4, 3).$

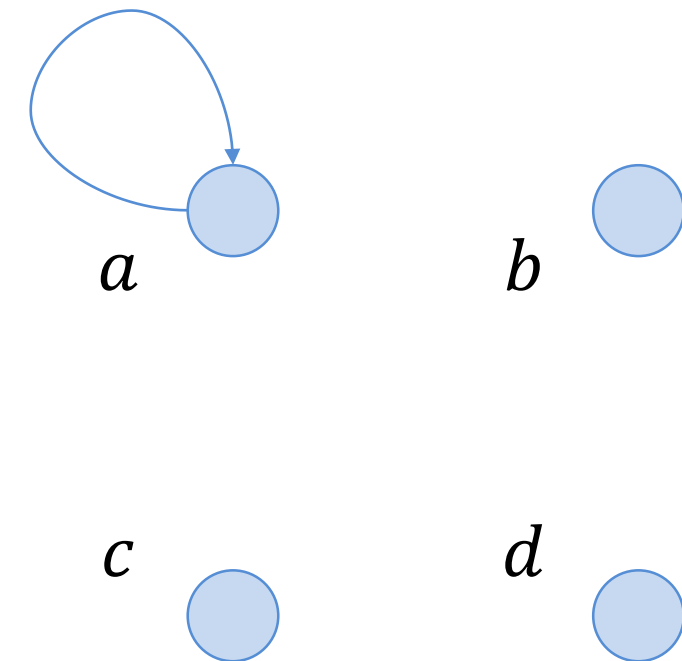


Determining which Properties a Relation has from its Digraph

- **Reflexivity**: A loop must be present at all vertices in the graph.
- **Irreflexivity**: No loop can be present at each vertices in the graph.
- **Symmetry**: If (x, y) is an edge, then so is (y, x) .
 - A symmetric relation can be represented by an undirected graph.
- **Antisymmetry**: If (x, y) with $x \neq y$ is an edge, then (y, x) is not an edge.
- **Transitivity**: If (x, y) and (y, z) are edges, then so is (x, z) .

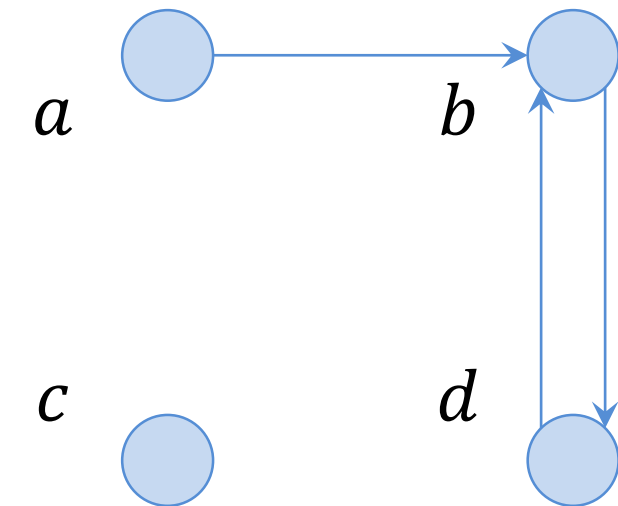
Determining which Properties a Relation has from its Digraph – Example 1

- Reflexive? No, not every vertex has a loop.
- Irreflexive? No, there is a loop (a, a) .
- Symmetric? Yes (trivially), there is no edge from one vertex to another.
- Antisymmetric? Yes (trivially), there is no edge from one vertex to another.
- Transitive? Yes, (trivially) since there is no edge from one vertex to another.



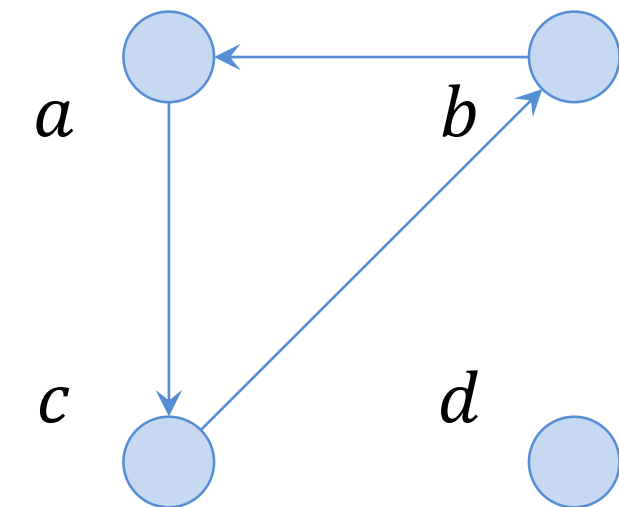
Determining which Properties a Relation has from its Digraph – Example 2

- Reflexive? No, there are no loops.
- Irreflexive? Yes, there are no loops.
- Symmetric? No, there is an edge from a to b , but not from b to a .
- Antisymmetric? No, there is an edge from d to b and b to d .
- Transitive? No, there are edges from b to d and from d to b , but there is no edge from d to d .



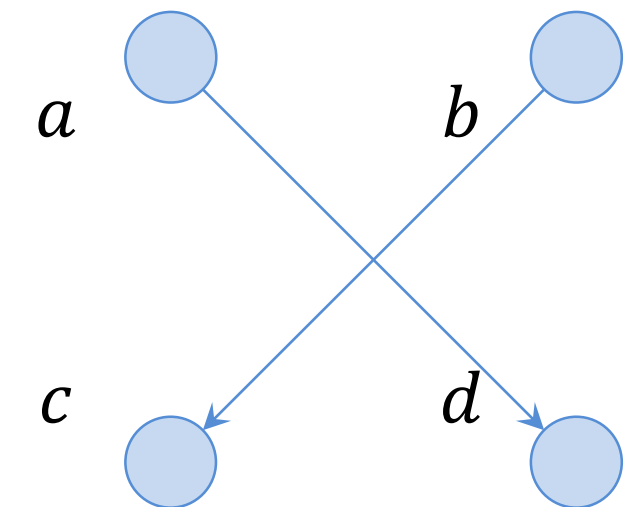
Determining which Properties a Relation has from its Digraph – Example 3

- Reflexive? No, there are no loops.
- Irreflexive? Yes, there are no loops.
- Symmetric? No, for example, there is no edge from c to a .
- Antisymmetric? Yes, whenever there is an edge from one vertex to another, there is not one going back.
- Transitive? No, there is no edge from a to b .

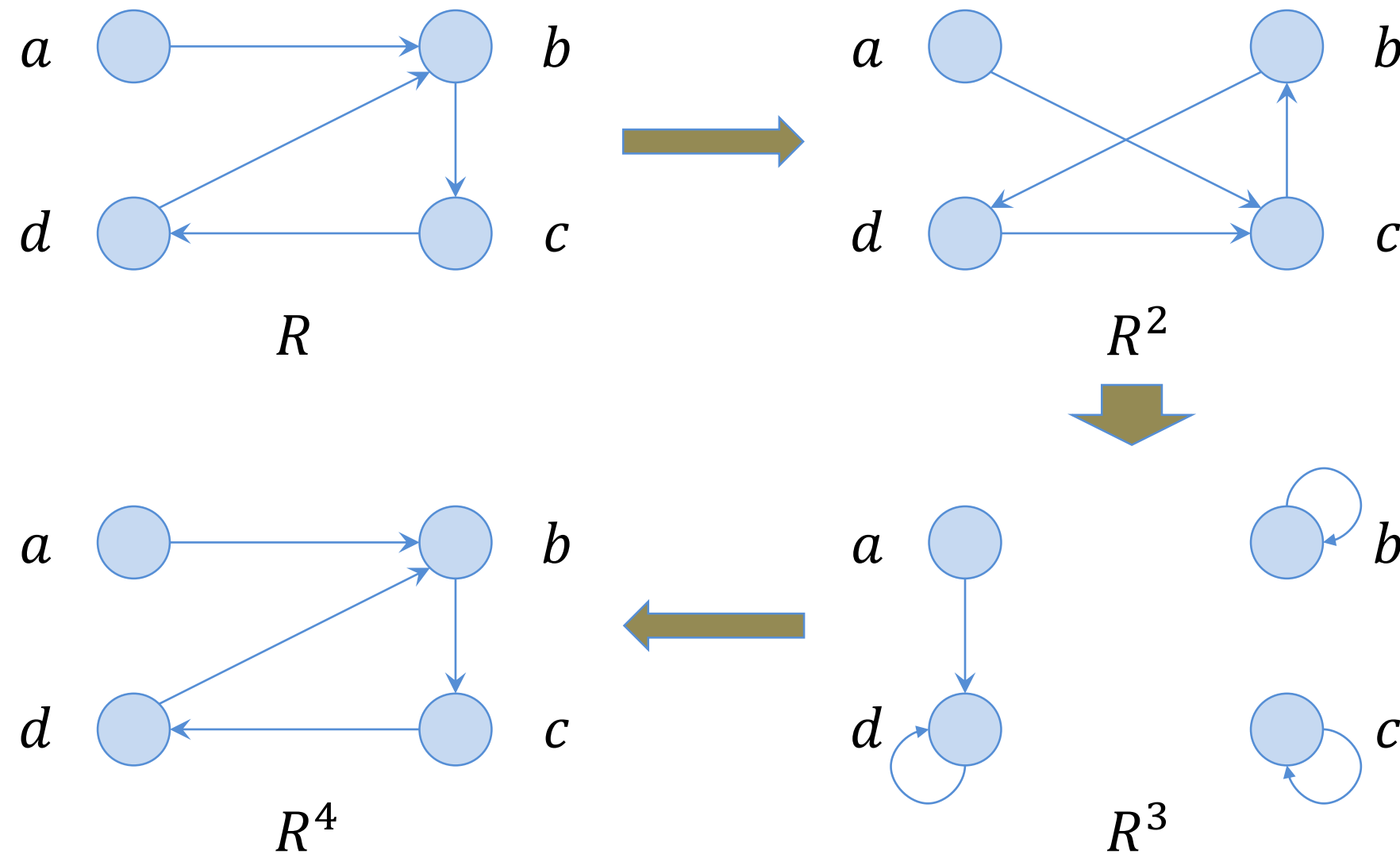


Determining which Properties a Relation has from its Digraph – Example 4

- Reflexive? No, there are no loops.
- Irreflexive? Yes, there are no loops.
- Symmetric? No, for example, there is no edge from d to a .
- Antisymmetric? Yes, whenever there is an edge from one vertex to another, there is not one going back.
- Transitive? Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins.



Example of the Powers of a Relation



- The pair (x, y) is in R^n if there is a path of length n from x to y in R (following the direction of the arrows).

CLOSURES OF RELATIONS

Section 9.4

Closure of R

Definition: Let R be a relation on a set A . R may or may not have some property P , such as reflexivity, symmetry, or transitivity. If there is a relation S with property P containing R such that S is a subset of every relation with property P containing R , then S is called the **closure** of R with respect to P . (Note that the closure of a relation with respect to a property may not exist.)

Reflexive Closure $r(R)$

- The **reflexive closure** of R equals $R \cup \Delta$, where $\Delta = \{(a, a) | a \in A\}$ is the diagonal relation on A .

Example: What is the reflexive closure of the relation $R = \{(a, b) | a < b\}$ on the set of integers?

Solution: The reflexive closure of R is

$$R \cup \Delta = \{(a, b) | a < b\} \cup \{(a, a) | a \in \mathbf{Z}\} = \{(a, b) | a \leq b\}$$

Symmetric Closure $s(R)$

- The **symmetric closure** of a relation can be constructed by taking the union of a relation with its inverse; that is, $R \cup R^{-1}$ is the symmetric closure of R , where $R^{-1} = \{(b, a) | (a, b) \in R\}$.

Example: What is the symmetric closure of the relation $R = \{(a, b) | a > b\}$ on the set of positive integers?

Solution: The symmetric closure of R is the relation

$$R \cup R^{-1} = \{(a, b) | a > b\} \cup \{(b, a) | a > b\} = \{(a, b) | a \neq b\}.$$

Paths in Directed Graphs

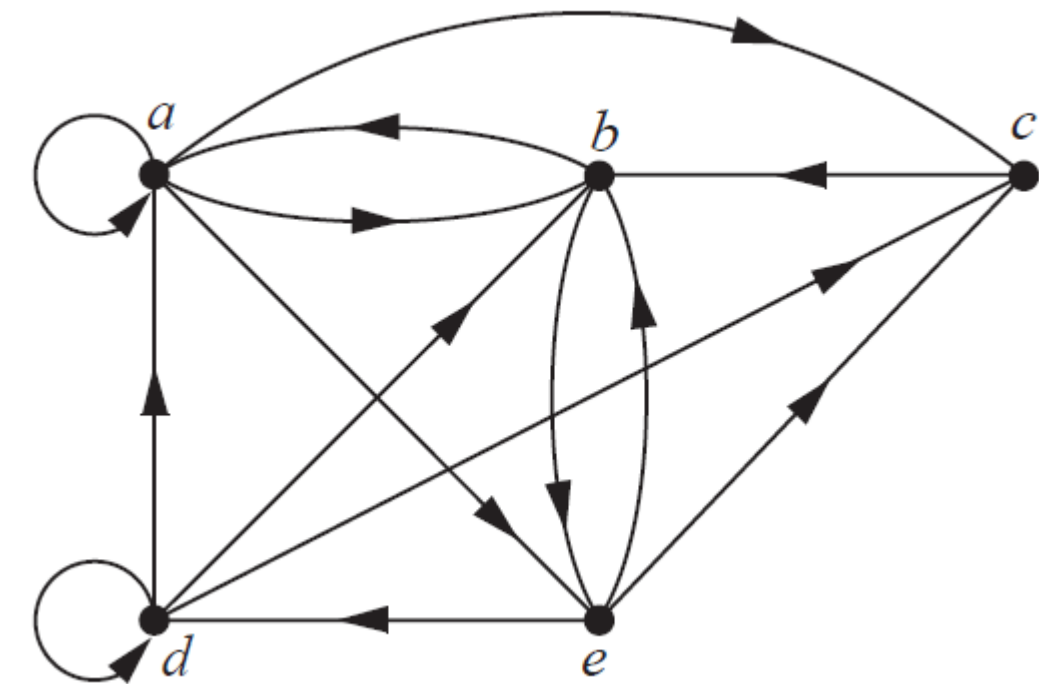
Definition: A **path** from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n)$ in G , where n is a nonnegative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1, x_2, \dots, x_{n-1}, x_n$ and has **length n** . We view the empty set of edges as a path of length zero from a to a . A path of length $n \geq 1$ that begins and ends at the same vertex is called a **circuit** or **cycle**.

- A path in a directed graph can pass through a vertex more than once. Moreover, an edge in a directed graph can occur more than once in a path.

Paths in Directed Graphs

Example: Which of the following are paths in the directed graph: a, b, e, d ; a, e, c, b ; b, a, c, b, a, a, b ; d, c ; c, b, a ; e, b, a, b, a, b, e ? What are the lengths of those that are paths? Which of the paths in this list are circuits?

The term **path** also applies to relations. Carrying over the definition from directed graphs to relations, there is a **path** from a to b in R if there is a sequence of elements $a, x_1, x_2, \dots, x_{n-1}, b$ with $(a, x_1) \in R$, $(x_1, x_2) \in R$, ..., and $(x_{n-1}, b) \in R$.



Path in a Relation

Theorem: Let R be a relation on a set A . There is a path of length n , where n is a positive integer, from a to b if and only if $(a, b) \in R^n$.

Proof: We will use mathematical induction. By definition, there is a path from a to b of length one if and only if $(a, b) \in R$, so the theorem is true when $n = 1$.

Assume that the theorem is true for the positive integer n . There is a path of length $n + 1$ from a to b if and only if there is an element $c \in A$ such that there is a path of length one from a to c , so $(a, c) \in R$, and a path of length n from c to b , that is, $(c, b) \in R^n$.

Consequently, there is a path of length $n + 1$ from a to b if and only if there is an element c with $(a, c) \in R$ and $(c, b) \in R^n$. But there is such an element if and only if $(a, b) \in R^{n+1}$.

Therefore, there is a path of length $n + 1$ from a to b if and only if $(a, b) \in R^{n+1}$.

Transitive Closures $t(R)$

Definition: Let R be a relation on a set A . The **connectivity relation** R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R .

- Because R^n consists of the pairs (a, b) such that there is a path of length n from a to b , it follows that R^* is the union of all the sets R^n . In other words,

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

- The connectivity relation is useful in many models.

Example: Let R be the relation on the set of all people in the world that contains (a, b) if a has met b . What is R^n , where n is a positive integer greater than one? What is R^* ?

Transitive Closures $t(R)$

Theorem: The transitive closure of a relation R equals the connectivity relation R^* .

Proof: Note that R^* contains R by definition. To show that R^* is the transitive closure of R , we must also show that R^* is transitive and that $R^* \subseteq S$ whenever S is a transitive relation that contains R .

First, we show that R^* is transitive. If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c in R^* . We obtain a path from a to c . Hence, $(a, c) \in R^*$. It follows that R^* is transitive.

Transitive Closures $t(R)$

Now suppose that S is a transitive relation containing R . Because S is transitive, S^n also is transitive and $S^n \subseteq S$ (by Theorem of Section 9.1). Furthermore, because $S^* = \bigcup_{k=1}^{\infty} S^k$ and $S^k \subseteq S$, it follows that $S^* \subseteq S$. Now note that if $R \subseteq S$, then $R^* \subseteq S^*$, because any path in R is also a path in S . Consequently, $R^* \subseteq S^* \subseteq S$. Thus, **any transitive relation that contains R must also contain R^*** . Therefore, R^* is the transitive closure of R .

Transitive Closures $t(R)$

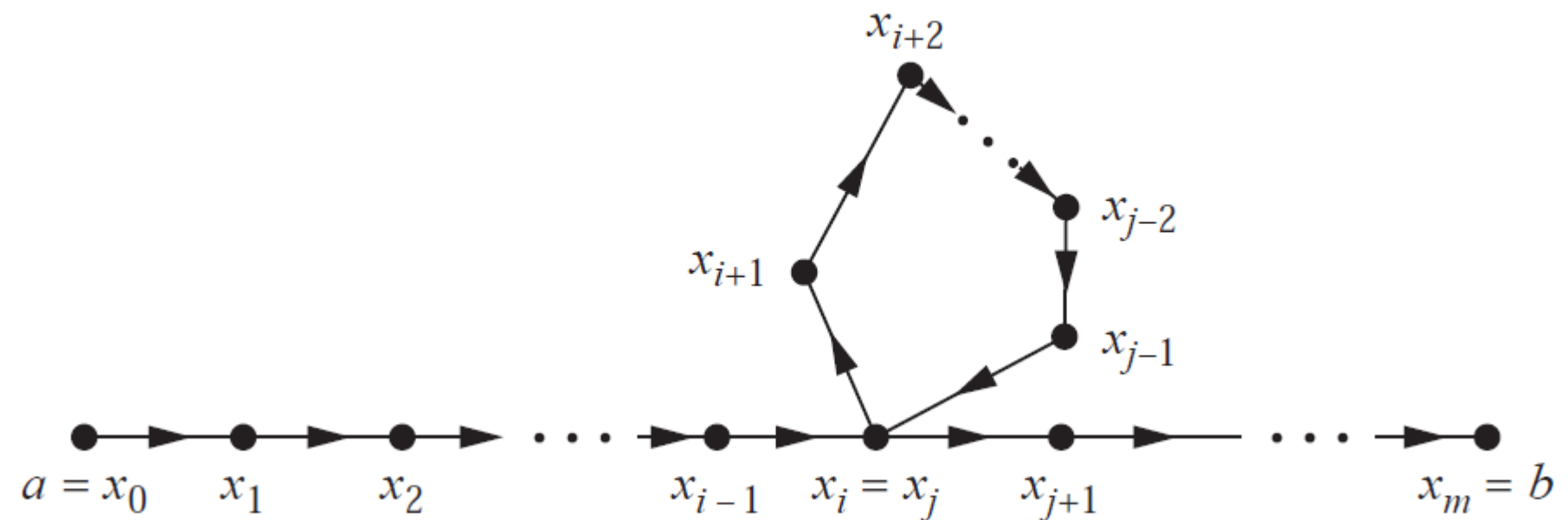
Lemma: Let A be a set with n elements, and let R be a relation on A . If there is a path of length at least one in R from a to b , then there is such a path with length not exceeding n . Moreover, when $a \neq b$, if there is a path of length at least one in R from a to b , then there is such a path with length not exceeding $n - 1$.

Proof: Suppose there is a path from a to b in R . Let m be the length of the shortest such path. Suppose that $x_0, x_1, x_2, \dots, x_{m-1}, x_m$, where $x_0 = a$ and $x_m = b$ is such a path.

Suppose that $a = b$ and that $m > n$, so that $m \geq n + 1$. By the pigeonhole principle, among the m vertices $x_0, x_1, x_2, \dots, x_{m-1}$, at least two are equal.

Transitive Closures $t(R)$

Suppose that $x_i = x_j$ with $0 \leq i < j \leq m - 1$. Then the path contains a circuit from x_i to itself. This circuit can be deleted from the path from a to b , leaving a path, namely, $x_0, x_1, \dots, x_i, x_{j+1}, \dots, x_{m-1}, x_m$ from a to b of shorter length. Hence, the path of shortest length must have length less than or equal to n .



$$R^* = R \cup R^2 \cup \dots \cup R^n$$

Transitive Closures $t(R)$

Theorem: Let M_R be the 0 – 1 matrix of the relation R on a set with n elements. Then the 0 – 1 matrix of the transitive closure R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee \cdots \vee M_R^{[n]}$$

Example: Find the 0 – 1 matrix of the transitive closure of the relation R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Transitive Closures $t(R)$

Solution:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad M_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad M_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

it follows that

$$\begin{aligned} M_{R^*} &= M_R \vee M_R^{[2]} \vee M_R^{[3]} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{aligned}$$

Algorithm1

Algorithm: A Procedure for Computing the Transitive Closure.

procedure *transitive closure* (M_R : 0 – 1 $n \times n$ matrix)

$A := M_R$

$B := A$

for $i := 2$ **to** n

$A := A \odot M_R$

$B := B \vee A$

return B { B is the 0 – 1 matrix for R^* }

The Number of Bit Operations of Algorithm1

- Computing the Boolean powers $M_R, M_R^{[2]}, \dots, M_R^{[n]}$ requires that $n - 1$ Boolean products of $n \times n$ 0 – 1 matrices be found. Each of these Boolean products can be found using $n^2(2n - 1)$ bit operations. Hence, these products can be computed using $n^2(2n - 1)(n - 1)$ bit operations.
- To find M_R^* from the n Boolean powers of M_R , $n - 1$ joins of 0 – 1 matrices need to be found. $(n - 1)n^2$ bit operations are used in this part of the computation.
- Therefore, the matrix of the transitive closure of a relation on a set with n elements can be found using $n^2(2n - 1)(n - 1) + (n - 1)n^2 = 2n^3(n - 1)$, which is $O(n^4)$ bit operations.
- We will describes a more efficient algorithm later.

Warshall's Algorithm

- If $a, x_1, x_2, \dots, x_{m-1}, b$ is a path, its interior vertices are $x_1, x_2, \dots, x_{m-1}$, that is, all the vertices of the path that occur somewhere other than as the first and last vertices in the path. The interior vertices of a, c, d, a, f, b are c, d, a, f .
- Warshall's algorithm is based on the construction of a sequence of 0 – 1 matrices. These matrices are $W_0, W_1, \dots, W_n$, where $W_0 = M_R$ is the 0 – 1 matrix of this relation, and $W_k = [w_{ij}^k]$. If there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2, \dots, v_k\}$ and is 0 otherwise.
- $W_n = M_{R^*}$

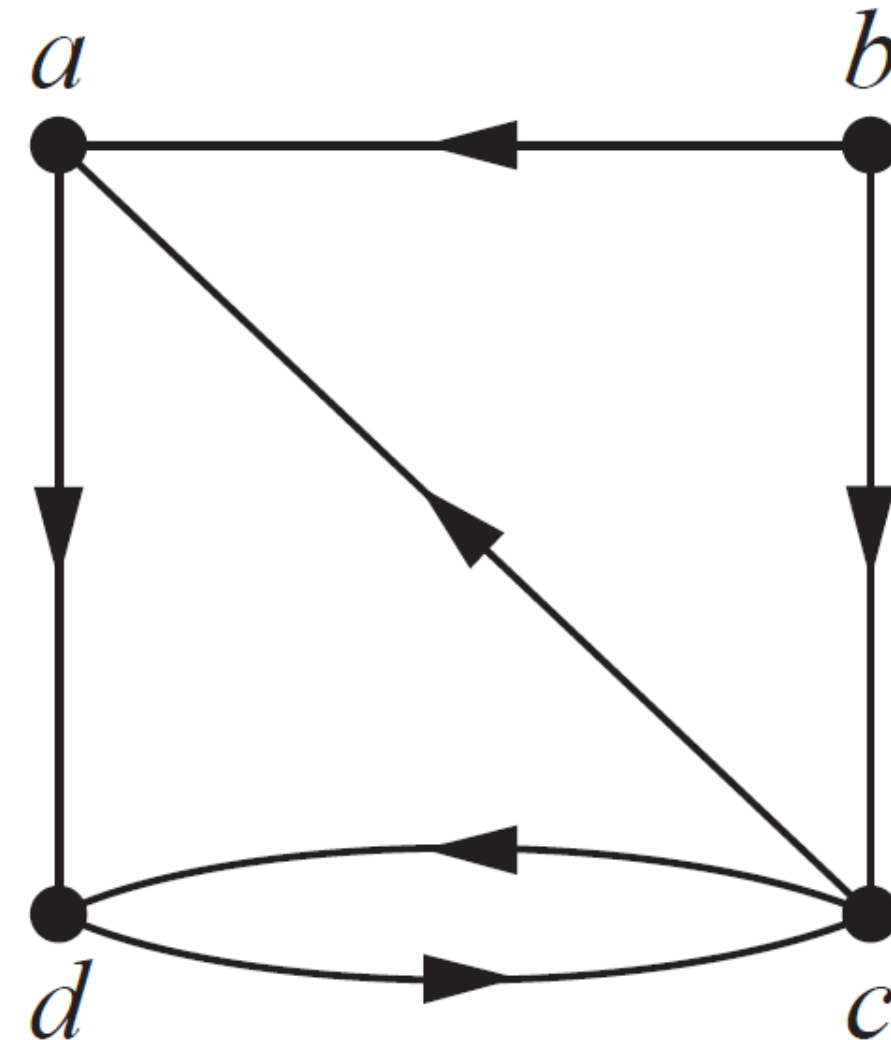
Warshall's Algorithm

Example: Let R be the relation with directed graph. Let a, b, c, d be a listing of the elements of the set. Find the matrices W_0, W_1, W_2, W_3 and W_4 . The matrix W_4 is the transitive closure of R .

Solution: Let $v_1 = a, v_2 = b, v_3 = c, v_4 = d$.

W_0 is the matrix of the relation. Hence,

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

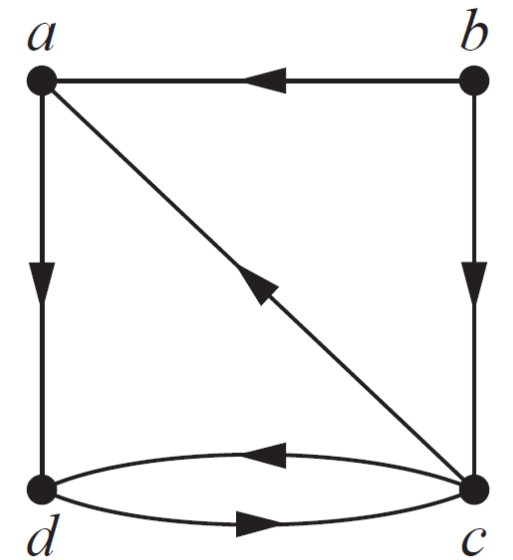


Warshall's Algorithm

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- W_1 has 1 as its (i, j) th entry if there is a path from v_i to v_j that has only $v_1 = a$ as an interior vertex. Note that all paths of length one can still be used because they have no interior vertices. There are paths *bad* and *cad*. Hence,

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



- W_2 has 1 as its (i, j) th entry if there is a path from v_i to v_j that has only $v_1 = a$ and/or $v_2 = b$ as its interior vertices, if any. Because there are no edges that have b as a terminal vertex, no new paths are obtained when we permit b to be an interior vertex. Hence, $W_2 = W_1$.

Warshall's Algorithm

$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- W_3 has 1 as its (i, j) th entry if there is a path from v_i to v_j that has only $v_1 = a, v_2 = b$, and/or $v_3 = c$ as interior vertices. We now have paths bca , bcd , dca and dcd .

Hence,

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

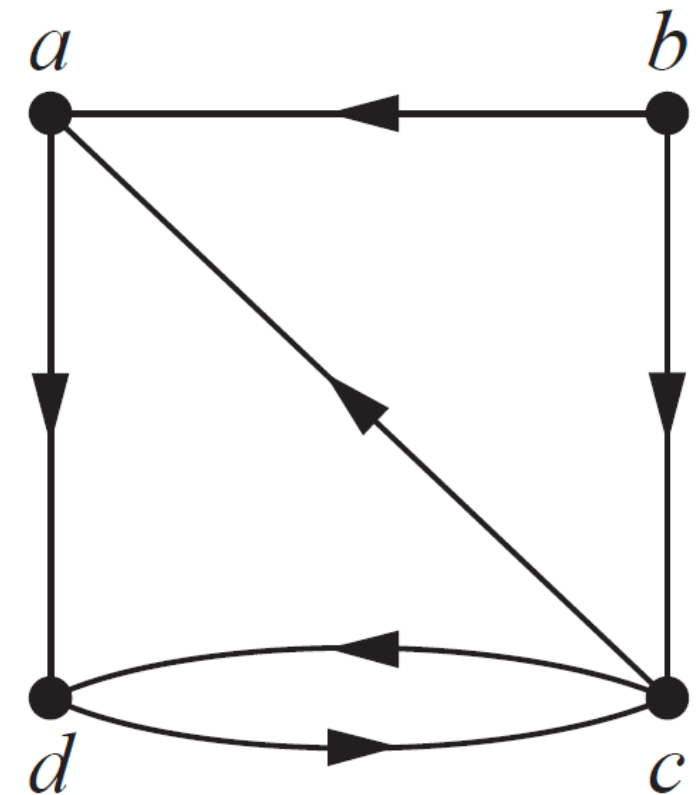
Warshall's Algorithm

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & \textcolor{red}{1} \\ 1 & 0 & 1 & \textcolor{red}{1} \\ 1 & 0 & 0 & \textcolor{red}{1} \\ 1 & 0 & 1 & \textcolor{red}{1} \end{bmatrix}$$

- Finally, W_4 has 1 as its (i, j) th entry if there is a path from v_i to v_j that has $v_1 = a, v_2 = b, v_3 = c$ and/or $v_4 = d$ as interior vertices. We now have paths $adca, adc, ad, bcdca, bc dc, bcd, cdca, cdc$ and cd .

$$W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

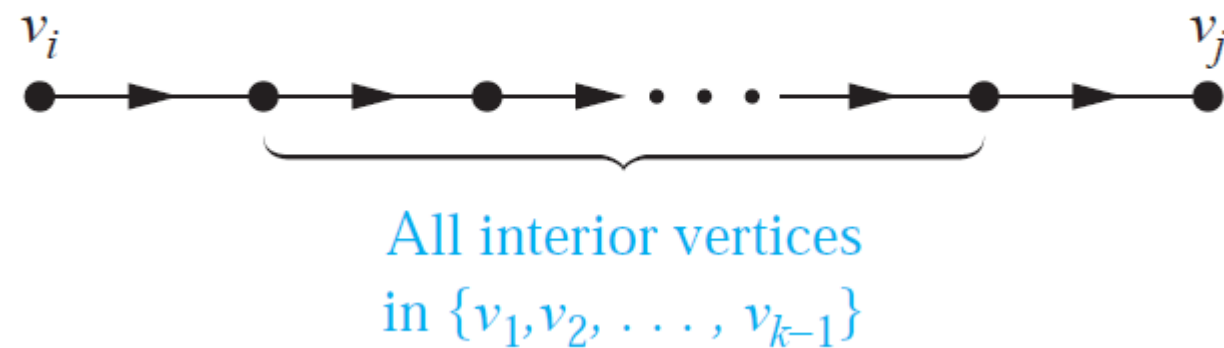
- W_4 is the matrix of the transitive closure.



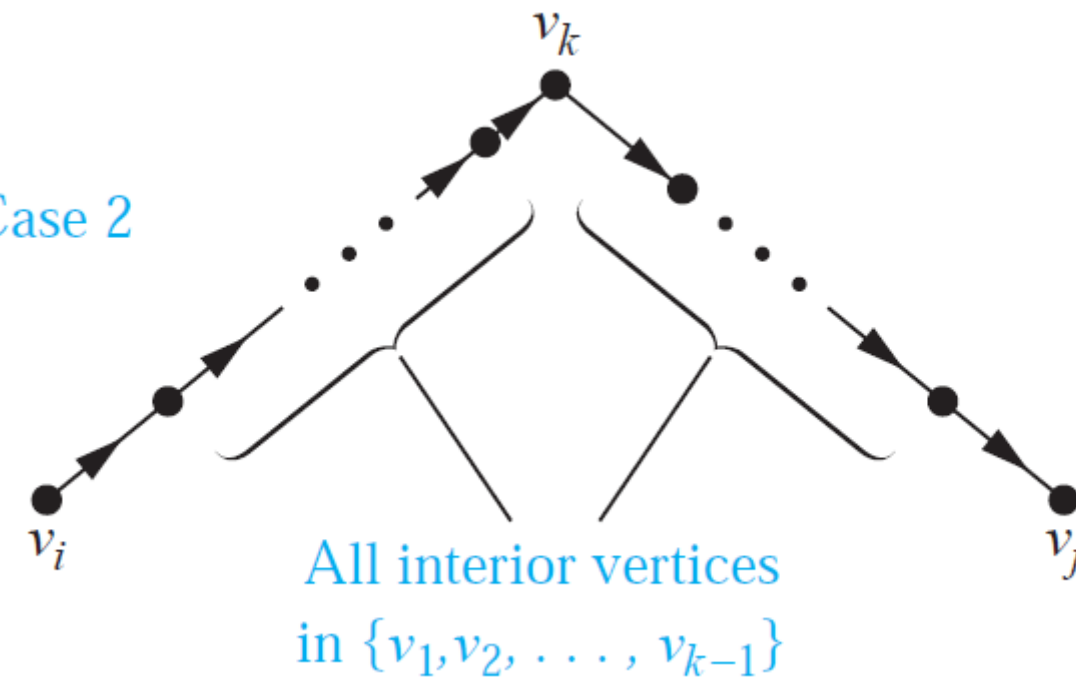
Compute W_k Directly from W_{k-1}

- There is a path from v_i to v_j with no vertices other than $v_1, v_2, \dots, v_k$ as interior vertices if and only if either there is a path from v_i to v_j with its interior vertices among the first $k - 1$ vertices in the list, or there are paths from v_i to v_k and from v_k to v_j that have interior vertices only among the first $k - 1$ vertices in the list.

Case 1



Case 2



Compute W_k Directly from W_{k-1}

- The first type of path exists if and only if $w_{ij}^{[k-1]} = 1$, and the second type of path exists if and only if both $w_{ik}^{[k-1]} = 1$ and $w_{kj}^{[k-1]} = 1$. Hence, $w_{ij}^{[k]} = 1$ if and only if either $w_{ij}^{[k-1]} = 1$ or both $w_{ik}^{[k-1]} = 1$ and $w_{kj}^{[k-1]} = 1$.

Lemma: Let $W_k = w_{ij}^{[k]}$ be the 0 – 1 matrix that has a 1 in its (i, j) th position if and only if there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \dots, v_k\}$.

Then

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee \left(w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]} \right)$$

whenever i, j and k are positive integers not exceeding n .

Warshall Algorithm

ALGORITHM2 Warshall Algorithm.

procedure Warshall (M_R : $n \times n$ 0 – 1 matrix)

$W := M_R$

for $k := 1$ **to** n

for $i := 1$ **to** n

for $j := 1$ **to** n

$w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$

Return $W \{W = [w_{ij}] \text{ is } M_{R^*}\}$

To find all n^2 entries of W_k from those of W_{k-1} requires $2n^2$ bit operations. Because Warshall's algorithm begins with $W_0 = M_R$ and computes the sequence of n 0 – 1 matrices $W_1, W_2, \dots, W_n = M_{R^*}$, the total number of bit operations used is $n \times 2n^2 = 2n^3$, which is $O(n^3)$ bit operations.

Closure of Closure of R

Theorem: $rs(R) = sr(R)$, $rt(R) = tr(R)$

Proof:

$$\begin{aligned} sr(R) &= s(R \cup \Delta) = (R \cup \Delta) \cup (R \cup \Delta)^{-1} \\ &= (R \cup \Delta) \cup (R^{-1} \cup \Delta^{-1}) = R \cup R^{-1} \cup \Delta = s(R) \cup \Delta = rs(R) \end{aligned}$$

$$\begin{aligned} tr(R) &= t(R \cup \Delta) = \bigcup_{i=1}^{\infty} (R \cup \Delta)^i = \bigcup_{i=1}^{\infty} \left(\bigcup_{j=1}^i R^j \cup \Delta \right) \\ &= \left(\bigcup_{i=1}^{\infty} R^i \right) \cup \Delta = t(R) \cup \Delta = rt(R) \end{aligned}$$

Closure of Closure of R

Theorem: $st(R) \subseteq ts(R)$

Proof:

$$st(R) = s\left(\bigcup_{i=1}^{\infty} R^i\right) = \left(\bigcup_{i=1}^{\infty} R^i\right) \cup \left(\bigcup_{i=1}^{\infty} R^i\right)^{-1} = \left(\bigcup_{i=1}^{\infty} R^i\right) \cup \left(\bigcup_{i=1}^{\infty} (R^{-1})^i\right) = \bigcup_{i=1}^{\infty} (R^i \cup (R^{-1})^i)$$

$$ts(R) = t(R \cup R^{-1}) = \bigcup_{i=1}^{\infty} (R \cup R^{-1})^i = \bigcup_{i=1}^{\infty} \left(R^i \cup (R^{-1})^i \cup \left(\bigcup_{k=1}^{i-1} (R^k \circ (R^{-1})^{i-k}) \right) \right)$$

EQUIVALENCE RELATIONS

Section 9.5

Section Summary

Equivalence
Relations

Equivalence
Classes

Equivalence
Classes and
Partitions

Equivalence Relations

Definition: A relation on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

Definition: Two elements a and b that are related by an equivalence relation are called **equivalent**. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Example: Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. R is reflexive, symmetric, and transitive. It follows that R is an equivalence relation.

Strings

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x . Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

- **Reflexivity:** Because $l(a) = l(a)$, it follows that aRa for all strings a .
- **Symmetry:** Suppose that aRb . Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa .
- **Transitivity:** Suppose that aRb and bRc . Since $l(a) = l(b)$, and $l(b) = l(c)$, $l(a) = l(c)$ also holds and aRc .

Congruence Modulo m

Example: Let m be an integer with $m > 1$. Show that the relation $R = \{(a, b) | a \equiv b \pmod{m}\}$ is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- **Reflexivity:** $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \times m$.
- **Symmetry:** Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m , and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)m$, so

$$b \equiv a \pmod{m}.$$

Congruence Modulo m

- **Transitivity**: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:

$$a - c = (a - b) + (b - c) = km + lm = (k + l)m.$$

Therefore,

$$a \equiv c \pmod{m}.$$

Divides

Example: Show that the “divides” relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is **not symmetric**. Hence, “divides” is not an equivalence relation.

- Reflexivity: $a|a$ for all a .
- Not Symmetric: For example, $2|4$ but $4 \nmid 2$. Hence, the relation is not symmetric.
- Transitivity: Suppose that a divides b and b divides c . Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c . Therefore, the relation is transitive.

Equivalence Classes

Definition: Let R be an equivalence relation on a set A . The set of all elements that are related to an element a of A is called the **equivalence class of a** . The equivalence class of a with respect to R is denoted by $[a]_R$.

- When only one relation is under consideration, we can write $[a]$, without the subscript R , for this equivalence class.

$$[a]_R = \{s | (a, s) \in R\}$$

Equivalence Classes

- If $b \in [a]_R$, then b is called a **representative** of this equivalence class. Any element of a class can be used as a representative of the class.
- The equivalence classes of the relation congruence modulo m are called the **congruence classes modulo m** . The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\dots, a - 2m, a - m, a + m, a + 2m, \dots\}$. For example,

$$[0]_4 = \{\dots, -8, -4, 0, 4, 8, \dots\}$$

$$[1]_4 = \{\dots, -7, -3, 1, 5, 9, \dots\}$$

$$[2]_4 = \{\dots, -6, -2, 2, 6, 10, \dots\}$$

$$[3]_4 = \{\dots, -5, -1, 3, 7, 11, \dots\}$$

Equivalence Classes and Partitions

Theorem: let R be an equivalence relation on a set A . These statements for elements a and b of A are equivalent:

(1) aRb

(2) $[a] = [b]$

(3) $[a] \cap [b] \neq \emptyset$

Proof: We first show that (1) implies (2). Assume that aRb . Now suppose that $c \in [a]$. Then aRc . Because aRb and R is symmetric, bRa . Because R is transitive and bRa and aRc , bRc . Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$. A similar argument shows that $[b] \subseteq [a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we have shown that $[a] = [b]$.

Equivalence Classes and Partitions

Second, we will show that (2) implies (3). Assume that $[a] = [b]$. It follows that $[a] \cap [b] \neq \emptyset$ because $[a]$ is nonempty ($a \in [a]$ because R is reflexive).

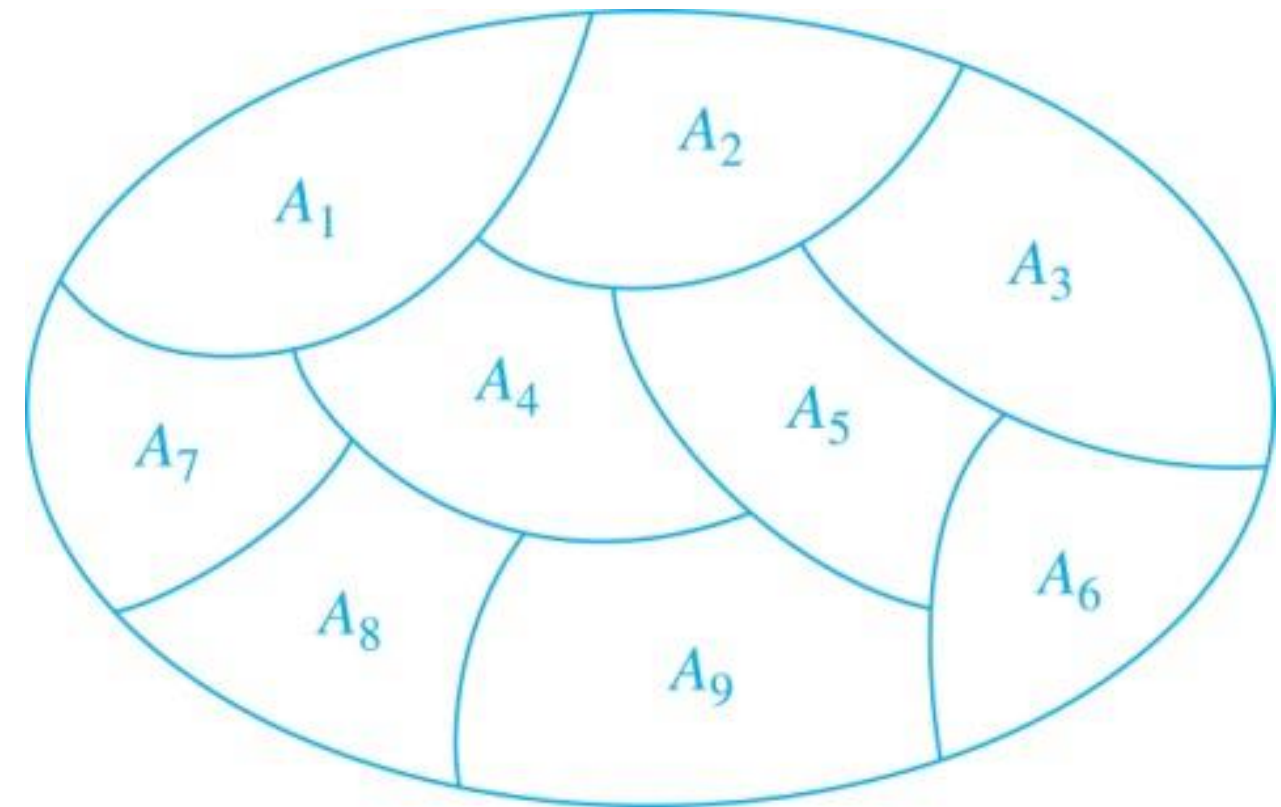
Next, we will show that (3) implies (1). Suppose that $[a] \cap [b] \neq \emptyset$. Then there is an element c with $c \in [a]$ and $c \in [b]$. In other words, aRc and bRc . By the symmetric property, cRb . Then by transitivity, because aRc and cRb , we have aRb .

Because (1) implies (2), (2) implies (3), and (3) implies (1), the three statements, (1), (2) and (3) are equivalent.

Partition of a Set

Definition: A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i , where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $\bigcup_{i \in I} A_i = S$.



An Equivalence Relation Partitions a Set

- Let R be an equivalence relation on a set A . The union of all the equivalence classes of R is all of A , since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A$$

- These equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \emptyset$ when $[a]_R \neq [b]_R$.
- Therefore, the equivalence classes form a partition of A , because they split A into disjoint subsets.

An Equivalence Relation Partitions a Set

Theorem: Let R be an equivalence relation on a set S . Then the equivalence classes of R form a partition of S . Conversely, given a partition $\{A_i | i \in I\}$ of the set S , there is an equivalence relation R that has the sets $A_i, i \in I$ as its equivalence classes.

Proof: We have already shown the first part of the theorem.

For the second part, assume that $\{A_i | i \in I\}$ is a partition of S . Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

An Equivalence Relation Partitions a Set

- Reflexivity: For every $a \in S$, $(a, a) \in R$, because a is in the same subset as itself.
- Symmetry: If $(a, b) \in R$, then b and a are in the same subset of the partition, so $(b, a) \in R$.
- Transitivity: If $(a, b) \in R$ and $(b, c) \in R$, then a and b are in the same subset of the partition, as are b and c . Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a, c) \in R$ since a and c belong to the same subset of the partition.

PARTIAL ORDERINGS

Section 9.6

Section Summary

Partial Orderings
and Partially-
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Partial Orderings

Definition: A relation R on a set S is called a **partial ordering**, or **partial order**, if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering R is called a **partially ordered set**, or **poset**, and is denoted by (S, R) . Members of S are called **elements** of the poset.

Example: Show that the “greater than or equal” relation ($\geq$) is a partial ordering on the set of integers.

- **Reflexivity:** $a \geq a$ for every integer a .
- **Antisymmetry:** If $a \geq b$ and $b \geq a$, then $a = b$.
- **Transitivity:** If $a \geq b$ and $b \geq c$, then $a \geq c$.

Partial Orderings

Example: Show that the divisibility relation ($|$) is a partial ordering on the set of integers.

- **Reflexivity:** $a|a$ for all integers a .
 - **Antisymmetry:** If a and b are positive integers with $a|b$ and $b|a$, then $a = b$.
 - **Transitivity:** Suppose that a divides b and b divides c . Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c .
Therefore, the relation is transitive.
- $(\mathbb{Z}^+, |)$ is a poset.

Partial Orderings

Example: Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S .

- **Reflexivity:** $A \subseteq A$ whenever A is a subset of S .
 - **Antisymmetry:** If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
 - **Transitivity:** If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
- The notation $a \preceq b$ is used to denote that $(a, b) \in R$ in an arbitrary poset (S, R) .

Comparability

Definition: The elements a and b of a poset $(S, \preceq)$ are **comparable** if either $a \preceq b$ or $b \preceq a$. When a and b are elements of S so that neither $a \preceq b$ nor $b \preceq a$, then a and b are called **incomparable**.

Definition: If $(S, \preceq)$ is a poset and every two elements of S are comparable, S is called a **totally ordered** or **linearly ordered set**, and $\preceq$ is called a **total order** or a **linear order**. A totally ordered set is also called a **chain**.

Definition: $(S, \preceq)$ is a **well-ordered set** if it is a poset such that $\preceq$ is a total ordering and every nonempty subset of S has a least element.

The Principle of Well-ordered Induction

Theorem: Suppose that S is a well-ordered set. Then $P(x)$ is true for all $x \in S$, if

INDUCTIVE STEP: for every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x < y$, then $P(y)$ is true.

Proof: Suppose it is not the case that $P(x)$ is true for all $x \in S$. Then there is an element $y \in S$ such that $P(y)$ is false. Consequently, the set $A = \{x \in S \mid P(x) \text{ is false}\}$ is nonempty. Because S is well ordered, A has a least element a , $P(x)$ is true for all $x \in S$ with $x < a$. By the inductive step $P(a)$ is true. This is a contradiction.

Remark: We do not need a basis step in a proof using the principle of well-ordered induction because if x_0 is the least element of a well ordered set, there are no elements $x \in S$ with $x < x_0$, so that $P(x)$ is true for all $x \in S$ with $x < x_0$.

Lexicographic Order

Definition: Given two posets $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ the **lexicographic ordering** on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2) that is, $(a_1, a_2) \prec (b_1, b_2)$, either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$.

- This definition can be easily extended to a lexicographic ordering on strings.

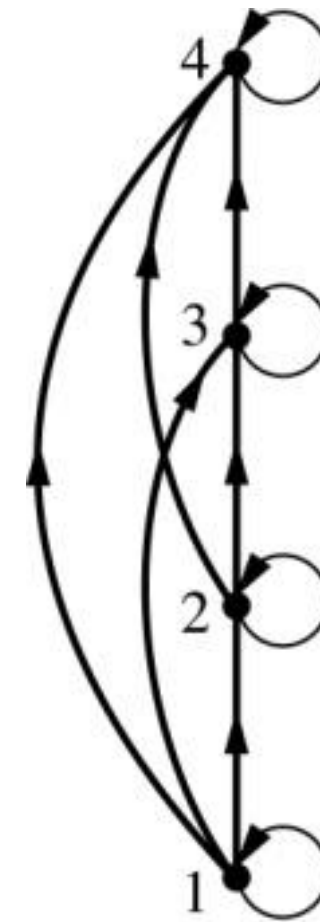
Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- *discre^et* $\prec$ *discrete^t*, because these strings differ in the seventh position and $e \prec t$.
- *discreet* $\prec$ *discreetⁿess*, because the second string is longer.

Hasse Diagrams

Definition: A **Hasse diagram** is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.

A partial ordering is shown in (a) of the figure above. The loops due to the reflexive property are deleted in (b). The edges that must be present due to the transitive property are deleted in (c). The Hasse diagram for the partial ordering (a), is depicted in (c).



(a)



(b)



(c)

Procedure for Constructing a Hasse Diagram

- To represent a finite poset $(S, \preceq)$ using a Hasse diagram, start with the directed graph of the relation:
 - Remove the loops (a, a) present at every vertex due to the reflexive property.
 - Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
 - Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

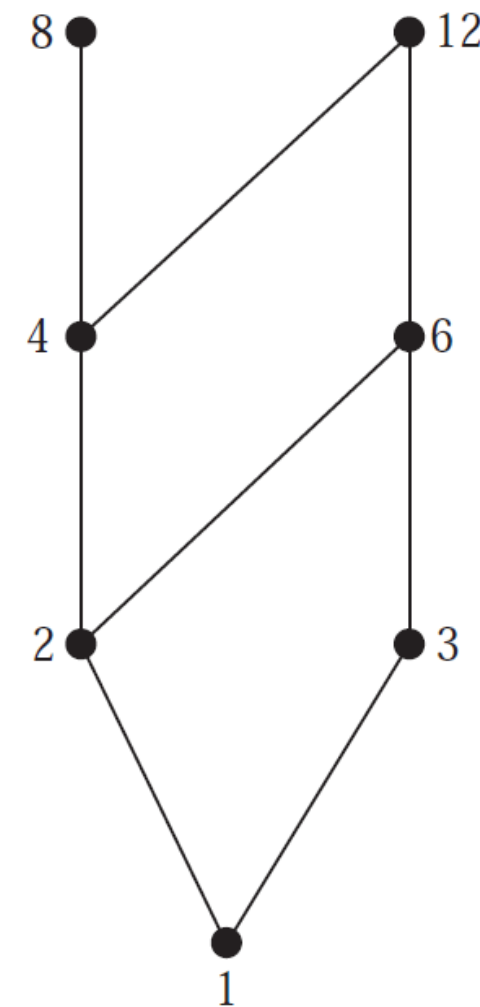
Covering Relation

Definition: Let $(S, \preceq)$ be a poset. We say that an element $y \in S$ **covers** an element $x \in S$ if $x \prec y$ and there is no element $z \in S$ such that $x \prec z \prec y$. The set of pairs (x, y) such that y covers x is called the **covering relation** of $(S, \preceq)$.

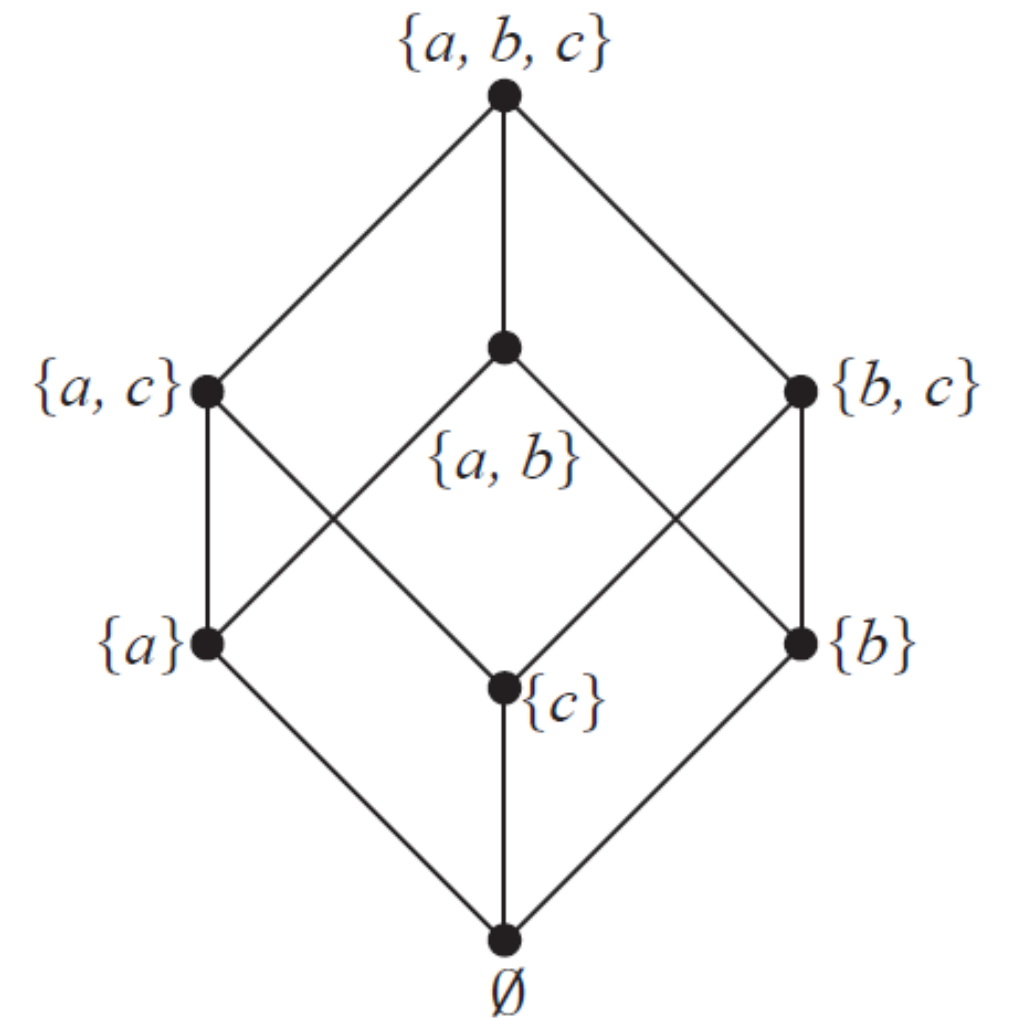
- We can recover a poset from its covering relation, because it is the reflexive transitive closure of its covering relation.
- This tells us that we can construct a partial ordering from its Hasse diagram.

Hasse Diagrams

Example: Draw the Hasse diagram representing the partial ordering $\{(a, b) | a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.



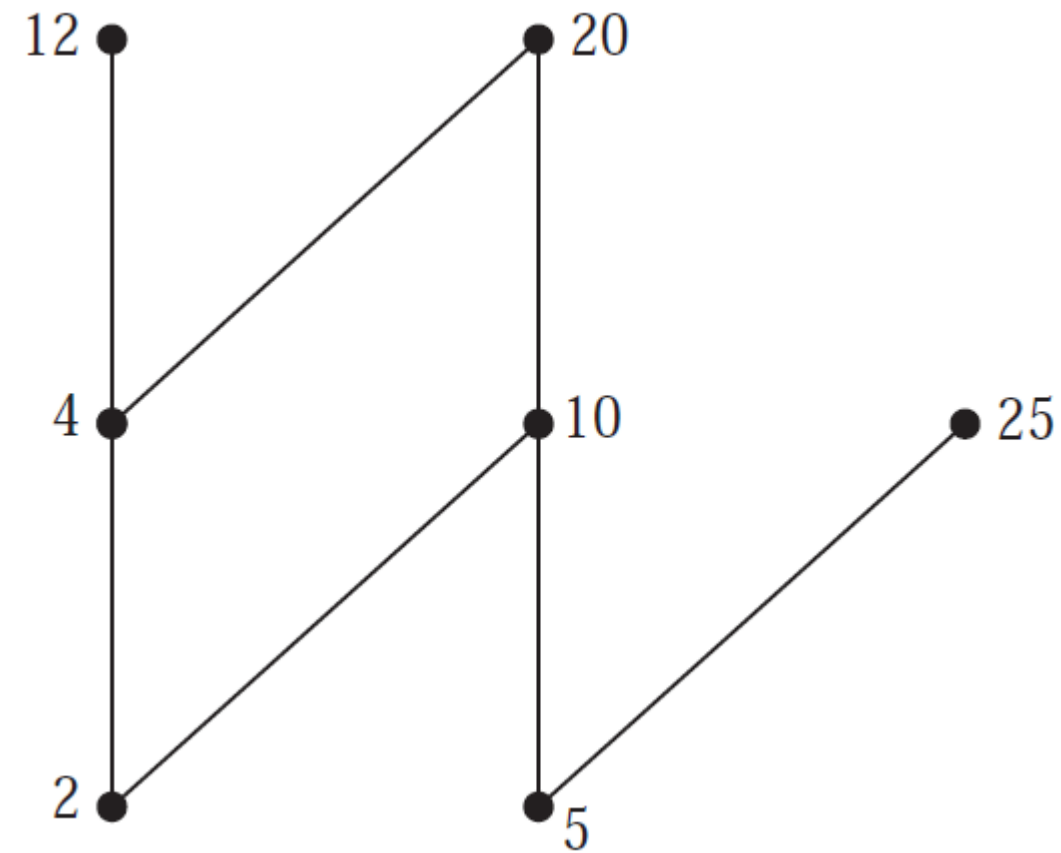
Example: Draw the Hasse diagram for the partial ordering $\{(A, B) | A \subseteq B\}$ on the power set $\mathcal{P}(S)$ where $S = \{a, b, c\}$.



Maximal and Minimal Elements

- a is **maximal** in the poset $(S, \leq)$ if there is no $b \in S$ such that $a < b$.
- a is **minimal** in the poset $(S, \leq)$ if there is no $b \in S$ such that $b < a$.

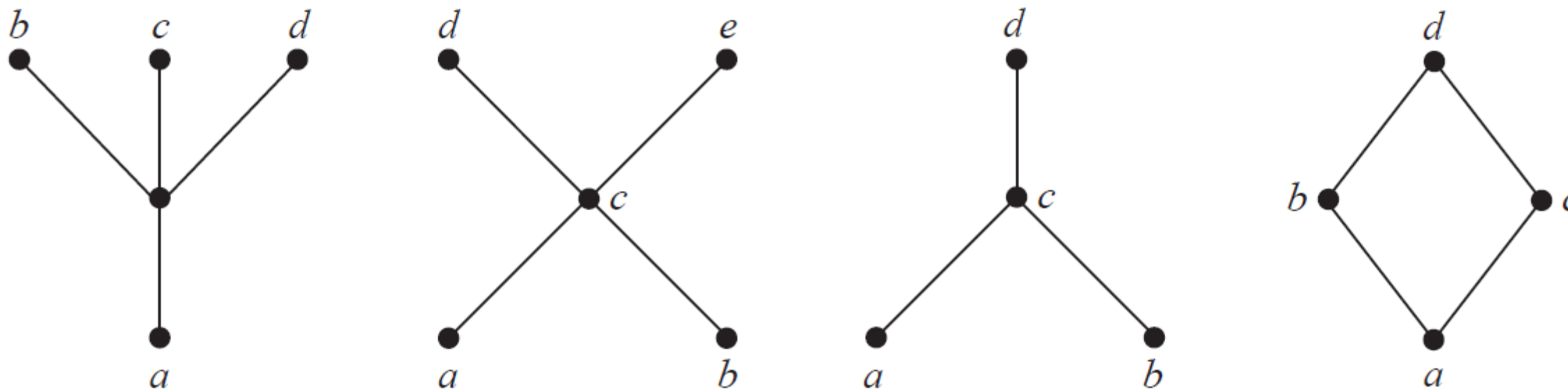
Example: Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal, and which are minimal?



Greatest and Least Element

- a is the **greatest element** of the poset $(S, \preceq)$ if $b \preceq a$ for all $b \in S$.
- a is the **least element** of the poset $(S, \preceq)$ if $a \preceq b$ for all $b \in S$.

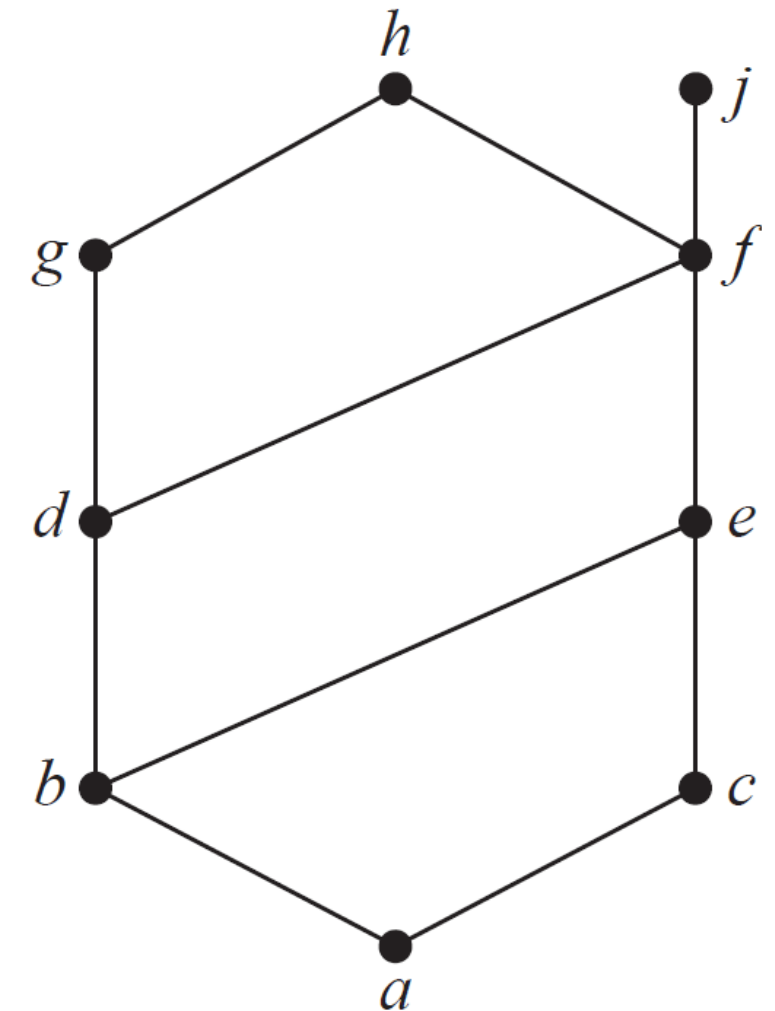
Example: Determine whether the posets represented by each of the Hasse diagrams have a greatest element and a least element.



Example: Let S be a set. Determine whether there is a greatest element and a least element in the poset $(\mathcal{P}(S), \subseteq)$.

Upper Bound and Lower Bound

- u is called an **upper bound** of A if u is an element of S such that $a \preceq u$ for all $a \in A$.
- l is called a **lower bound** of A if l is an element of S such that $l \preceq a$ for all $a \in A$.
- x is called the **least upper bound** of the subset A if x is an upper bound that is less than every other upper bound of A , denoted by **$\text{lub}(A)$** .
- y is called the **greatest lower bound** of the subset A if y is a lower bound of A and $z \preceq y$ whenever z is a lower bound of A , denoted by **$\text{glb}(A)$** .



Lattices

Definition: A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**.

Example: Is the poset $(\mathbf{Z}^+, |)$ a lattice?

Solution: Let a and b be two positive integers. The least upper bound and greatest lower bound of these two integers are the least common multiple and the greatest common divisor of these integers. It follows that this poset is a lattice.

Example: Determine whether $(\mathcal{P}(S), \subseteq)$ is a lattice where S is a set.

Solution: Let A and B be two subsets of S . The least upper bound and the greatest lower bound of A and B are $A \cup B$ and $A \cap B$. Hence, $(\mathcal{P}(S), \subseteq)$ is a lattice.

Topological Sorting

Definition: A total ordering $\preceq$ is said to be **compatible** with the partial ordering R if $a \preceq b$ whenever aRb . Constructing a compatible total ordering from a partial ordering is called **topological sorting**.

Lemma: Every finite nonempty poset $(S, \preceq)$ has at least one minimal element.

Proof: Choose an element a_0 of S . If a_0 is not minimal, then there is an element a_1 with $a_1 \prec a_0$. If a_1 is not minimal, there is an element a_2 with $a_2 \prec a_1$. Continue this process, so that if a_n is not minimal, there is an element a_{n+1} with $a_{n+1} \prec a_n$. Because there are only a finite number of elements in the poset, this process must end with a minimal element a_n .

Topological Sorting

- The topological sorting algorithm we will describe works for any finite nonempty poset.

To define a total ordering on the poset $(A, \preceq)$, first choose a minimal element a_1 . Next, note that $(A - \{a_1\}, \preceq)$ is also a poset. If it is nonempty, choose a minimal element a_2 of this poset. Then remove a_2 as well. Continue this process by choosing a_{k+1} to be a minimal element in $A - \{a_1, a_2, \dots, a_k\}$, as long as elements remain.

Because A is a finite set, this process must terminate. The end product is a sequence of elements $a_1, a_2, \dots, a_n$. The desired total ordering $<_t$ is defined by

$$a_1 <_t a_2 <_t \dots <_t a_n$$

This total ordering is compatible with the original partial ordering.

Topological Sorting

Algorithm Topological Sorting

procedure topological sort $((S, \preceq)$: finite poset)

$k := 1$

while $S \neq \emptyset$

$a_k :=$ a minimal element of S {such an element exists by Lemma}

$S := S - \{a_k\}$

$k := k + 1$

return $a_1, a_2, \dots, a_n$ { $a_1, a_2, \dots, a_n$ is a compatible total ordering of S }

Topological Sorting

Example: Find a compatible total ordering for the poset $(\{1,2,4,5,12,20\}, |)$.

Minimal element chosen 1	5	2	4	20	12